

Ultraviolet divergences in non-renormalizable supersymmetric theories

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Abstract

We present a pedagogical review of our current understanding of the ultraviolet structure of $\mathcal{N} = (1,1)$ 6D supersymmetric Yang-Mills theory and of $\mathcal{N} = 8$ 4D supergravity. These theories are not renormalizable, they involve power ultraviolet divergences and, in all probability, an infinite set of higher-dimensional counterterms that contribute to on-mass-shell scattering amplitudes.

A specific feature of supersymmetric theories (especially, of extended supersymmetric theories) is that these counterterms may not be invariant off shell under the full set of supersymmetry transformations. The lowest-dimensional nontrivial counterterm is supersymmetric on shell. Still higher counterterms may lose even the on-shell invariance. On the other hand, the full effective Lagrangian, generating the amplitudes and representing an infinite sum of counterterms, still enjoys the complete symmetry of original theory.

We also discuss simple supersymmetric quantum-mechanical models that exhibit the same behaviour.

1 Introduction

The standard Einstein gravity and its supersymmetric extensions involve a dimensionful constant and are not renormalizable theories. Still, several years ago, certain hopes were expressed in the conference talks and in the literature that the extended $\mathcal{N} = 8$ Poincaré supergravity might be a “finite theory”, meaning that it does not involve relevant higher-dimensional counterterms.

To a considerable extent, these hopes were based on the explicit calculations [1], which demonstrated the absence of logarithmic divergences in on-mass-shell scattering amplitudes through 4 loops. This remarkable cancellation is explained by very high symmetry of the theory. This symmetry simply does not allow one to write down the counterterms of dimension¹ $d = 4, 6, 8, 10, 12, 14$ which enjoy on shell the general covariance, the extended $\mathcal{N} = 8$ supersymmetry and different dualities, and which could give rise to divergences.

However, most experts believe nowadays that, at the level $d = 16$ (corresponding to 7 loops) or, at any rate, at the level $d = 18$ (corresponding to 8 loops), such counterterms do appear and generate divergences.²

$\mathcal{N} = 8$ supergravity is a very complicated theory. That is why it is interesting to study a much simpler model, the $\mathcal{N} = (1, 1)$ 6D supersymmetric Yang-Mills (SYM) theory. Its coupling constant also carries dimension, so that the theory is not renormalizable. Nontrivial counterterms generating logarithmic divergences in the amplitude already appear there at the 3-loop level. The presence of these divergences was confirmed by explicit perturbative calculations in [2].

As was already mentioned in the Abstract, non-renormalizable theories with extended supersymmetry differ from the well-known non-renormalizable non-supersymmetric theories (Einstein’s gravity, Fermi theory of weak interactions, chiral theory describing low-energy pion physics) by the fact that extended supersymmetry cannot be kept *off shell* for higher-dimensional counterterms. Only the full effective Lagrangian, representing an infinite sum of counterterms of higher and higher dimension, keeps this symmetry. This fact is known to experts, but there is no clearly written and not too technical text where it would be pedagogically explained. That is what we will try to do in this lecture.³

We start in Sect. 2 by saying few words about generic features of non-supersymmetric non-renormalizable theories: chiral theory, Fermi theory and Einstein’s gravity. The main observation: one can in principle carry out the renormalization program and get rid of ultraviolet divergences also in a non-renormalizable theory, redefining order by order an infinite set of couplings, but it does not help one to calculate scattering amplitudes at the energies exceeding the *unitary limit* — an intrinsic feature of all non-renormalizable theories.

A general wisdom is that complicated field-theoretical phenomena can be much better understood by studying toy quantum-mechanical models having similar features. In Sect. 3, we consider the simplest supersymmetric quantum-mechanical (SQM) model, involving only

¹The word *dimension* will be used in this paper in two meaning. First, it is space-time dimension which is denoted by the capital D . Second, it is dimension of various operators which we will denote by the low-case d .

²We will discuss it later, but we hasten to say right now that all the nontrivialities mentioned above concern the *logarithmic* divergences. A theory with a dimensionful constant necessarily has *power* ultraviolet divergences, which are not associated to higher-dimensional counterterms. The calculations in [1] are not sensitive to power divergences.

³A reader interested in more technical issues should consult the recent paper [3], the logics of which we mostly follow here.

one real supervariable

$$X = x + \theta\bar{\psi} + \psi\bar{\theta} + F\theta\bar{\theta}. \quad (1.1)$$

We note in particular that the auxiliary field F can be algebraically excluded only in the simplest Witten's version of this model with the bosonic kinetic term $\propto \dot{x}^2$ [4]. If the Lagrangian involves higher derivatives, the field F becomes dynamical.

Generically, one cannot get rid of F in this case. However, if the action represents the sum of the Witten term and a higher derivative term, the field F can be formally excluded via an infinite perturbative series. The Lagrangian thus obtained involves only the fields $x, \psi, \bar{\psi}$ and their time derivatives of all orders. It is invariant under *modified* (compared to Witten's model) supersymmetry transformations that also include time derivatives of all orders. We discuss then the implications of this simple observation for $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ $4D$ SYM theories.

Sect. 4 is devoted to other instructive models. First, we discuss the so-called *maximally supersymmetric* SQM model. It is obtained by dimensional reduction from $\mathcal{N} = 1$ $10D$ SYM theory and has 16 real supercharges. The model involves Abelian flat directions or *vacuum valleys*, where the non-Abelian field strength and the associated potential vanish. In the bottom of this valley, far enough from the origin, the dynamics of the model is described by the effective low-energy Born-Oppenheimer (BO) Hamiltonian, representing an infinite series over the small BO parameter γ . The same is true for the corresponding Lagrangian,⁴

$$L^{\text{eff}} = L_0 + \gamma^3 L_1 + \gamma^6 L_2 + \dots \quad (1.2)$$

The leading term has a simple form and is invariant (up to a time derivative) under supersymmetry transformations that also have a simple form. The whole series is invariant under the modified transformations, which also represent an infinite series in γ ,

$$\delta = \delta_0 + \gamma^3 \delta_1 + \gamma^6 \delta_2 + \dots \quad (1.3)$$

But an individual term $L_{n \geq 1}$ in (1.2) is *not* supersymmetric. On the other hand, one can observe that L_1 is invariant under the same supersymmetry transformations as L_0 if the dynamical variables satisfy the equations of motion following from L_0 . This is the SQM counterpart of on-shell supersymmetry of the counterterms in $6D$ SYM and $4D$ supergravity.

At the end of Sect. 4, we go back to non-supersymmetric chiral field theory and study the structure of its *tree* amplitudes. We consider the leading term (2.1) of the chiral Lagrangian and observe that, in close analogy with (1.2), it can be represented as an infinite series of the terms of growing dimension, such that an individual term in this series does not have the full $SU_L(2) \times SU_R(2)$ symmetry of (2.1). The next-to leading term in this expansion enjoys, however, the full chiral symmetry *on shell* — in exactly the same way as the term L_1 in the series (1.2) enjoys on shell the full extended supersymmetry.

Note that still higher terms $L_{2,3,\dots}$ in this expansion need not to be on-shell symmetric. The scattering amplitudes keep, however, complete chiral symmetry at all orders.

In Sect. 5, we discuss $6D$ SYM theories, both $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ versions thereof. The $\mathcal{N} = (1, 0)$ theory involves a chiral anomaly that breaks gauge invariance. The $\mathcal{N} = (1, 1)$ theory is anomaly-free, but is not renormalizable because of dimensionful coupling. We briefly describe the harmonic superfield formalism, allowing one to understand

⁴We will see in Sect. 3 that, with a natural definition of γ , the series goes over γ^{3n} .

the symmetry structure of these theories in the most clear way and to write down explicit closed expressions for the actions.

In Sect. 6, we discuss the structure of higher-dimensional counterterms. The relevant counterterms appear at the 3-loop level, having canonical dimension $d = 10$. They are invariant under $\mathcal{N} = (1, 1)$ supersymmetry transformations only on shell, but not off shell. One can write two different such counterterms. Surprisingly, the presence of only one of them was “observed” in explicit loop calculations. The reason for such an unexpected cancellation is not clear yet.

Finally, in Sect. 7 we briefly discuss the situation in supergravity. Logarithmic UV divergences and associated counterterms probably appear there at the 7-loop or 8-loop level. In supergravity, total cross sections diverge, and it is difficult to define simple observables whose energy dependence could be studied. But the presence of higher-dimensional counterterms will definitely prohibit crossing the Planck mass barrier and performing meaningful perturbative calculations at high energies.

2 Non-supersymmetric theories

Historically, the first non-renormalizable field theory model that attracted the attention of theorists was Fermi’s 4-fermion model characterized by the dimensionful constant G_F . But we choose to discuss in some more detail the effective chiral theory developed in Ref. [5] and used thereafter for many practical calculations in low-energy QCD.

To leading order and neglecting u and d quark masses (so that pions are also massless), the effective chiral Lagrangian describing pion interactions reads

$$\mathcal{L}^{(0)} = \frac{F_\pi^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle, \quad (2.1)$$

where

$$U(x) = \exp \left\{ \frac{i\pi^a(x)\sigma^a}{F_\pi} \right\} \quad (2.2)$$

is an $SU(2)$ matrix, $\pi^{a=1,2,3}(x)$ are the pion fields, $F_\pi = 93$ MeV is the pion decay constant and $\langle \dots \rangle$ stands for the trace.

The Lagrangian possesses $SU_L(2) \times SU_R(2)$ symmetry — it is invariant under a multiplication of U on an arbitrary unitary matrix on the left or on the right.⁵ Expanding the exponential, one can derive,

$$\mathcal{L}^{(0)} = \frac{1}{2}(\partial_\mu \pi^a)^2 + \frac{1}{6F_\pi^2} [(\pi^a \partial_\mu \pi^a)^2 - (\pi^a \pi^a)(\partial_\mu \pi^b)^2] + \dots \quad (2.3)$$

The quartic term in the Lagrangian involves derivatives, and the $\pi\pi$ scattering amplitude grows with energy,

$$M_{\pi\pi \rightarrow \pi\pi}^{(0)} \sim \frac{E^2}{F_\pi^2}. \quad (2.4)$$

⁵There is also a $SU_L(3) \times SU_R(3)$ version of Eq.(2.1) describing interactions of the mesons of the pseudoscalar octet, but we do not need to discuss it.

This model is not renormalizable and involves power divergences, which show up in the loops.

People are often not concerned about power divergences, because the latter do not arise when the dimensional regularization (technically, the most simple method to calculate the Feynman graphs) is used. But to disregard them completely amounts to hiding the problem under the carpet. The best and the most physical, to our mind, regularization scheme is the lattice regularization. Anyway, to attribute a meaning to the path integral symbol, one should discretize the space-time and define the path integral as a continuum limit of a finite-dimensional integral.⁶ Using lattice regularization for non-renormalizable theories exhibits power divergences [6]. The same is true for Slavnov's higher-derivative regularization scheme used usually for gauge theories [7]. Power divergences also appear for renormalizable theories including interacting scalar fields: recall the fine tuning problem in the Standard Model and the hierarchy problem in non-supersymmetric models of Grand Unification.

On top of power divergences, there might also be logarithmic divergences. In chiral theory, they already show up at the 1-loop level. The one-loop contribution to the amplitude reads

$$M_{\pi\pi\rightarrow\pi\pi}^{(1)} = \frac{\Lambda^2}{F_\pi^2} M_{\pi\pi\rightarrow\pi\pi}^{(0)} + \frac{A(s, t)}{F_\pi^4} \ln \frac{\Lambda}{\mu} + \frac{B(s, t, \mu)}{F_\pi^4} \quad (2.5)$$

with an arbitrarily chosen μ , on which B also logarithmically depends.

The last UV-finite term in (2.5) is nonlocal and complicated. The first two terms are local, however. Indeed, to obtain an ultraviolet divergence, the characteristic loop momenta should be large, and then a complicated loop graph is effectively shrunk to a point. We arrive at the notion of the *Wilsonian effective Lagrangian* that generates local contributions to the scattering amplitudes. The amplitude $M_{\pi\pi\rightarrow\pi\pi}^{(0)}$ is generated by (2.1) (and, at the 1-loop level, the coupling constant F_π^2 is renormalized, including power divergences), while the contribution $\sim A(s, t)$ is generated by two different higher-dimensional *counterterms*,

$$\mathcal{L}_1^{(1)} = \langle \partial^\mu U^\dagger \partial_\mu U \rangle^2, \quad \mathcal{L}_2^{(1)} = \langle \partial_\mu U^\dagger \partial_\nu U \rangle \langle \partial^\mu U^\dagger \partial^\nu U \rangle. \quad (2.6)$$

Different contributions to the amplitude are schematically represented in Fig. 1.

Note that one could, in principle, write down several other counterterms involving four derivatives. But these extra terms vanish when the field U satisfies the tree equations of motion,

$$(\square U)U^\dagger - U(\square U^\dagger) = 0, \quad (2.7)$$

As a result, these extra counterterms do not contribute to the *on-shell* scattering amplitudes and are irrelevant.⁷

Both in renormalizable and non-renormalizable theories, one can in principle get rid of all UV divergences, if hiding them in the constants of the bare Lagrangian, so that the

⁶This works for conventional field theories in flat space-time. How to define path integral for gravity is a separate difficult question that we are not going to address here.

⁷One can always get rid of the counterterms proportional to the tree equations of motion by redefining the fields. Schematically,

$$\mathcal{L}(\Phi) + \frac{\delta \mathcal{L}}{\delta \Phi} A(\Phi) \approx \mathcal{L}(\Phi + A(\Phi)). \quad (2.8)$$

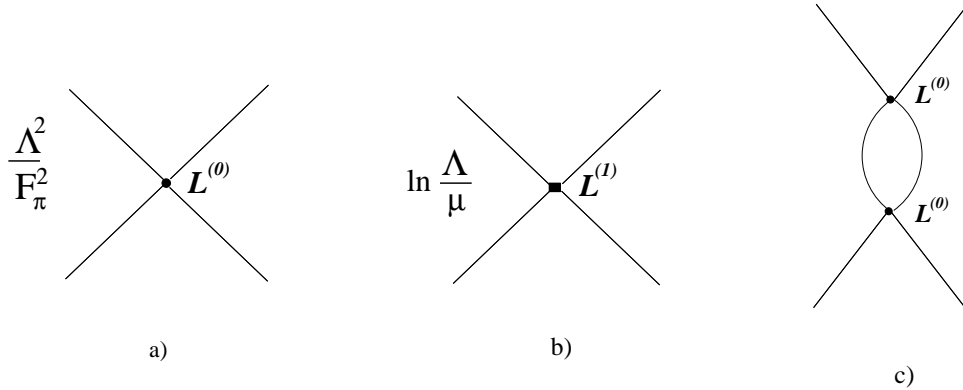


Figure 1: One-loop contributions to the pion scattering amplitudes. *a)* renormalization of F_π^{-2} , *b)* logarithmically divergent local contributions, *c)* UV-finite nonlocal contributions.

renormalized constants would be UV-finite. For a non-renormalizable theory, one should do it for an infinite number of the bare constants associated with the infinite number of relevant counterterms. Even though this procedure looks rather unnatural, it allows one to treat a non-renormalizable theory perturbatively, and not only at the tree level, but also with taking loops into account.

It is well-known, however, that such calculations make sense only at low enough energies. In non-renormalizable theories, tree amplitude rapidly grow with energy. For example, the tree pion scattering amplitude behaves as in (2.4). But this growth cannot keep going indefinitely. This would violate unitarity — cross sections cannot grow faster than $\propto \ln^2 E$.⁸ That means that, starting from the energy $E \sim F_\pi$ ⁹ (the *unitarity limit* for chiral theory), loop corrections to the tree scattering amplitudes become essential. Indeed, by dimensional counting, the second and the third terms in (2.5) grow with energy as $\sim E^4$. At $E \sim \mu_{\text{hadr}}$, they are of the same order as the tree amplitude (2.4). μ_{hadr} is the scale where *all* loop corrections become equally important and perturbative calculations are not possible anymore.

The author remembers well that, at the beginning of the 1970-ties, when most theorists did not look at the direction of the Standard Model yet, they tried to cross the unitarity barrier in Fermi theory by inventing ingenious resummation schemes [9]. But this did not work. Actually, there is no much sense to calculate loops with the Fermi Lagrangian even at the energies below the unitary limit, because it is practically impossible to separate the loop corrections from the contribution due to higher-derivative counterterms, which enter with unknown arbitrary coefficients.

In effective chiral theory, loop calculations make, however, a certain sense. They bring about a nontrivial information because of the presence of so-called *chiral logarithms* - large factors $\sim \ln(\mu_{\text{hadr}}/m_\pi)$ associated with *infrared* divergences in massless theory, which contribute to strong amplitudes. But, again, this works only for low enough energies.

The conventional Einstein's gravity is also a non-renormalizable theory. The Einstein-

⁸One can prove it if the spectrum does not involve massless particles [8]. To apply this argument for the chiral theory, one should endow the pions with the mass by adding the term $\sim \langle U \rangle + \langle U^\dagger \rangle$ in the Lagrangian.

⁹Well, rather from $E \sim 2\pi F_\pi \sim \mu_{\text{hadr}}$.

Hilbert Lagrangian reads

$$\mathcal{L}^{(0)} = -\frac{1}{16\pi G_N} R. \quad (2.9)$$

Newton's constant $G_N = m_{Pl}^{-2}$ carries dimension, it plays exactly the same role as G_F in Fermi theory and F_π^{-2} in chiral theory. Tree graviton scattering amplitudes grow with energy.

Gravity has, however, a distinguishing feature, compared to two other theories. In gravity, one-loop graphs are *free* from logarithmic divergences (power divergences survive). Indeed, the logarithmic 1-loop divergences should be associated with the appearance of the counterterms of canonical dimension $d = 4$. The only such general covariant structures are

$$\mathcal{L}_1^{(1)} = R^2, \quad \mathcal{L}_2^{(1)} = R_{\mu\nu} R^{\mu\nu} \quad (2.10)$$

[the square of the Riemann tensor $R_{\mu\nu\alpha\beta}$ is expressed via a linear combination of the structures (2.10) plus a total derivative]. One can observe, however, that these structures *vanish* on the mass shell — for the fields satisfying the Einstein equations of motion in empty space, $R_{\mu\nu} = 0$. And hence the logarithmic 1-loop divergences cancel [10].

One can often hear people saying that “Einstein's gravity is finite at one loop”. One should clearly understand, however, that they mean thereby only the absence of *logarithmic* ultraviolet divergences (the absence of the analog of the second term in (2.5) for graviton scattering amplitudes). But power divergences, similar to those that show up in the first term of (2.5), are still there. And the statement that, after renormalization, the 1-loop contribution to the amplitude is of the same order as the tree contribution, when the energies are of order of the Planck mass, is still there.

Going back to logarithmic ultraviolet divergences, they reappear in gravity at the 2-loop level. There exists a counterterm of canonical dimension $d = 6$, which does not vanish on mass shell,

$$\mathcal{L}^{(2)} = R_{\mu\nu\alpha\beta} R^{\mu\nu\gamma\delta} R^{\alpha\beta}_{\gamma\delta}. \quad (2.11)$$

This brings about the contribution $\propto G_N^3 \ln \Lambda$ in the graviton-graviton scattering amplitude.

In other words, the Einstein-Hilbert Lagrangian, like Fermi's 4-fermion Lagrangian and like the pion interaction Lagrangian (2.1), describes well low-energy phenomena, but general relativity is an effective, rather than the fundamental theory. We will argue later that the same probably concerns Poincaré supergravity and extended Poincaré supergravity. No calculations are possible in these theories beyond the Planck mass barrier.

3 Witten's model with higher derivatives

In non-supersymmetric models, the counterterms have the same symmetry as the tree Lagrangian — chiral symmetry for the effective model of pion interactions and general covariance for gravity. However, the situation is different in supersymmetric theories — more often than not, one cannot keep the full symmetry of the model in interest for individual counterterms of a given canonical dimension. This behaviour can be best understood by studying certain toy SQM models [3].

The simplest possible example is Witten's SQM system involving one bosonic degree of freedom $x(t)$ and its fermionic superpartners $\psi(t)$ and $\bar{\psi}(t)$. The Lagrangian of the model

reads

$$L_0 = \frac{\dot{x}^2 - [V'(x)]^2}{2} + \frac{i}{2} (\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}) + V''(x)\bar{\psi}\psi. \quad (3.1)$$

The corresponding equations of motion are

$$\begin{aligned} \ddot{x} + V'(x)V''(x) - V'''(x)\bar{\psi}\psi &= 0, \\ i\dot{\psi} - V''(x)\psi &= 0, \\ i\dot{\bar{\psi}} + V''(x)\bar{\psi} &= 0. \end{aligned} \quad (3.2)$$

The Lagrangian (3.1) is invariant (up to a total time derivative) under the following nonlinear supersymmetry transformations

$$\begin{aligned} \delta x &\equiv \delta_\epsilon x + \delta_{\bar{\epsilon}} x = \epsilon\bar{\psi} + \psi\bar{\epsilon}, \\ \delta\psi &\equiv \delta_\epsilon\psi = -\epsilon[i\dot{x} + V'(x)], \\ \delta\bar{\psi} &\equiv \delta_{\bar{\epsilon}}\bar{\psi} = \bar{\epsilon}[i\dot{x} - V'(x)]. \end{aligned} \quad (3.3)$$

In (0+1) dimensions, there is no need of renormalization and of introducing higher-dimensional counterterms. Still, one can study relatives of the Lagrangian L_0 , having higher canonical dimension and involving higher time derivatives, on their own merits.

The primary observation is that it is *impossible* to write a Lagrangian depending on the fields $x, \psi, \bar{\psi}$ and involving their *higher* time derivatives which would be invariant under the transformations (3.3). This is due to the well-known fact that the Lie brackets of the transformations (3.3) do not close off mass shell, but only on mass shell. When acting on the variable $x(t)$, the Lie bracket $(\delta_{\bar{\epsilon}}\delta_\xi - \delta_\xi\delta_{\bar{\epsilon}})$ boils down to a total time derivative. But it is not so for the fermion variables. For example,

$$(\delta_{\bar{\epsilon}}\delta_\xi - \delta_\xi\delta_{\bar{\epsilon}})\psi = \xi\bar{\epsilon}[i\dot{\psi} + V''(x)\psi] = 2i\xi\bar{\epsilon}\dot{\psi} + \xi\bar{\epsilon}\frac{\partial L}{\partial\psi}. \quad (3.4)$$

The presence of the second term in (3.4) does not affect the invariance of L_0 under (3.3). Indeed,

$$(\delta_{\bar{\epsilon}}\delta_\xi - \delta_\xi\delta_{\bar{\epsilon}})L_0 = 2i\xi\bar{\epsilon}\dot{L}_0 + \xi\bar{\epsilon}\left(\frac{\partial L_0}{\partial\psi}\frac{\partial L_0}{\partial\bar{\psi}} + \frac{\partial L_0}{\partial\bar{\psi}}\frac{\partial L_0}{\partial\psi}\right) = 2i\xi\bar{\epsilon}\dot{L}_0. \quad (3.5)$$

But, for $L \neq L_0$, the second term in the Lie bracket $(\delta_{\bar{\epsilon}}\delta_\xi - \delta_\xi\delta_{\bar{\epsilon}})L$ vanishes only on the mass shell of L_0 .

The standard way to solve this problem and to construct fully supersymmetric actions of any dimension is to introduce a supervariable (1.1). The transformations of the superspace coordinates generate *linear* supersymmetry transformations of the dynamic variables,

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} + \psi\bar{\epsilon}, \\ \delta\psi &= \epsilon(F - i\dot{x}), \\ \delta\bar{\psi} &= \bar{\epsilon}(F + i\dot{x}), \\ \delta F &= i(\epsilon\dot{\bar{\psi}} - \dot{\psi}\bar{\epsilon}). \end{aligned} \quad (3.6)$$

Any higher-derivative action of the form

$$S = \int dt d\bar{\theta} d\theta \left(\frac{1}{2} \bar{D} X P \left[i \frac{\partial}{\partial t} \right] D X + V(X) \right), \quad (3.7)$$

where $P(\partial/\partial t)$ is an arbitrary polynomial and

$$D = \frac{\partial}{\partial \theta} + i\bar{\theta} \frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial \bar{\theta}} - i\theta \frac{\partial}{\partial t} \quad (3.8)$$

are the supersymmetric covariant derivatives, is invariant under (3.6).

The original Witten's model [4] did not involve higher time derivatives, $P(z) = 1$. In Ref. [11], interesting higher-derivative models with $P(z) = z$ and $P(z) = a + bz$ were analyzed. The component Lagrangian of the model with $P(z) = z$ reads

$$L = \dot{x}\dot{F} + \dot{\bar{\psi}}\dot{\psi} + V'(x)F + V''(x)\bar{\psi}\psi. \quad (3.9)$$

By construction, it is invariant modulo a total derivative under transformations (3.6). In contrast to the Lagrangian (3.1), the field F now enters with derivatives. It is dynamical, not auxiliary anymore.

The spectrum of the corresponding quantum Hamiltonian (it now involves two pairs of dynamical variables) is not bounded neither from above, nor from below. The absence of the ground state means the presence of *ghosts*. Generically, ghosts bring about *collapse*: the system runs into a singularity in a finite time (the same phenomenon as falling into the center for the attractive potential $V(r) = \alpha/r^2$ with large enough α), the probability “leaks through” and the unitarity is lost.

It turns out, however, that, in the particular model (3.9), the ghosts are “benign”: there is no collapse, the Hamiltonian is Hermitian (in spite of the absence of the ground state) and the unitarity is preserved [11, 12].

In [13], we conjectured that the fundamental Theory of Everything is not String Theory, but a conventional quantum field theory living in a flat higher-dimensional bulk (and our Universe represents a classical 3-brane solution of this theory). For renormalizability, this theory should involve higher time and spatial derivatives. Then it must involve ghosts, but the ghosts should be of benign variety: the Hamiltonian should be still Hermitian, and the S -matrix still unitary...

But in this paper, we are not interested in the dynamical properties of the higher-derivative models. It is the fact that one cannot get rid of the former auxiliary field F in this system which is of a principal importance for us now.

The much-studied four-dimensional supersymmetric gauge theories exhibit the same behaviour. Consider first the $\mathcal{N} = 1$, 4D supersymmetric SYM Lagrangian. It involves the gauge fields and gluinos and is invariant under certain nonlinear supersymmetry transformations. One also can write higher-derivative off-shell supersymmetric Lagrangians of canonical dimensions $d = 6, 8$, etc., but they necessarily include the auxiliary field D of the vector multiplet, which now becomes dynamical. In this case, supersymmetry is realized linearly.

The same is true for the $\mathcal{N} = 2$ supersymmetric SYM theory. We have a gauge superfield \mathcal{W} involving a triplet of auxiliary fields D^A . Higher-derivatives supersymmetric Lagrangians like $\mathcal{L} \sim \langle \int d^8\theta \mathcal{W}^2 \bar{\mathcal{W}}^2 \rangle$ can be written, and they involve the derivatives of D^A . For the “matter” fields belonging to the $\mathcal{N} = 2$ hypermultiplet, the full set of the auxiliary fields is

infinite. The latter can be presented as components of a certain $\mathcal{N} = 2$ harmonic superfield [14]. Higher-derivative off-shell-invariant actions can also be written in that case.

But for the $\mathcal{N} = 4$ theory, the situation is different. Superfield formalism, with all supersymmetries being manifest and off-shell, is not developed, the full set of auxiliary fields is not known and probably does not exist. Thus, one simply *cannot* write in this case an off-shell supersymmetric higher-derivative action.¹⁰

In many cases, one can write, however, complicated higher-derivative effective Lagrangians not involving auxiliary fields and fully off-shell supersymmetric. Again, this can be best understood by considering a simple SQM example. We now consider the model with $P(z) = 1 + gz^2$ [3]. In this case, the ghosts are malicious enough to bring about the collapse and to kill the Hermiticity. But, as we only use this as a toy model displaying the structure of the effective Lagrangians in complicated field theories of interest, we need not to worry about it. We obtain the following component Lagrangian,

$$L = \frac{1}{2}(\dot{x}^2 + F^2) + i\dot{\psi}\psi + FV'(x) + V''(x)\bar{\psi}\psi + g\frac{1}{2}\left(\ddot{x}^2 + \dot{F}^2 + 2i\ddot{\psi}\dot{\psi}\right). \quad (3.10)$$

This Lagrangian is invariant under the transformations (3.6). Also in this case the formerly auxiliary field F has become dynamical and cannot be eliminated algebraically. Still, one can now integrate out the field F perturbatively through the formal power series solution

$$F = -\sum_{n=0}^{\infty} g^n \frac{d^{2n}V'(x)}{dt^{2n}}. \quad (3.11)$$

One obtains in this way the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + g\ddot{x}^2) + i\dot{\psi}\psi + ig\ddot{\psi}\dot{\psi} - \frac{1}{2}\sum_{n=0}^{\infty} (-g)^n \left(\frac{d^n V'(x)}{dt^n}\right)^2 + V''(x)\bar{\psi}\psi, \quad (3.12)$$

which involves only x, ψ and $\bar{\psi}$ and is by construction invariant with respect to the nonlinear supersymmetry transformations,

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} + \psi\bar{\epsilon}, \\ \delta\psi &= \epsilon\left(-i\dot{x} - \sum_{n=0}^{\infty} g^n \frac{d^{2n}V'(x)}{dt^{2n}}\right), \quad \delta\bar{\psi} = \bar{\epsilon}\left(i\dot{x} - \sum_{n=0}^{\infty} g^n \frac{d^{2n}V'(x)}{dt^{2n}}\right), \end{aligned} \quad (3.13)$$

which close modulo the equations of motion for the full Lagrangian (3.12). For example,

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\psi = -2\epsilon_1\epsilon_2 \sum_{n=0}^{\infty} g^n \frac{d^{2n}}{dt^{2n}} \left(\frac{\partial L}{\partial\psi}\right). \quad (3.14)$$

The Lagrangian (3.12) represents an infinite perturbative series in g ,

$$L = \sum_{n=0}^{\infty} g^n L_n = L_0 + gL_1 + g^2L_2 + \dots, \quad (3.15)$$

¹⁰We make a terminological comment on what exactly do we mean by “off-shell”. A Lagrangian is called off-shell symmetric if the corresponding action is invariant under certain variations of dynamic variables, without imposing any supplementary conditions. In case of supersymmetry, the action is always off-shell invariant if it can be expressed in terms of appropriate superfields. But the inverse is not generally true. The $\mathcal{N} = 4$ 4D SYM action is off-shell invariant, but cannot be expressed into superfields.

where L_0 is written in (3.1). The same is true for the supersymmetry transformations (3.13): $\delta = \delta_0 + g\delta_1 + \dots$, with δ_0 written in (3.3).

The full variation of the full Lagrangian, δL , should represent a total time derivative. Disregarding the latter and expanding $\delta L \approx 0$ in g ,¹¹ we obtain in the first order in g ,

$$\delta_0 L_1 + \delta_1 L_0 \approx 0. \quad (3.16)$$

The variation of L_0 is proportional to the classical equations of motion (3.2). It follows that the first-order correction to the Lagrangian,

$$L_1 = \frac{1}{2}\ddot{x}^2 + i\ddot{\psi}\dot{\psi} + \frac{1}{2}\dot{x}^2(V''(x))^2, \quad (3.17)$$

is not invariant under the action of the nonlinear supersymmetry transformations (3.3) off shell, but it is invariant (modulo a total time derivative) *on shell*, when the equations of motion (3.2) are imposed as constraints.

On the contrary, the second-order correction

$$L_2 = -\frac{1}{2}[\ddot{x}V''(x) + \dot{x}^2V'''(x)]^2, \quad (3.18)$$

is not invariant with respect to δ_0 , but satisfies a more complicated condition,

$$\delta_0 L_2 + \delta_1 L_1 + \delta_2 L_0 \approx 0. \quad (3.19)$$

4 Two models

The example above was simple, but somewhat artificial. However, the situation when the effective Lagrangian represents an infinite series of higher-derivative terms, like in (3.12), and this Lagrangian is invariant under modified supersymmetry transformations, also representing an infinite series, is quite general. One known example is the non-Abelian Born-Infeld effective Lagrangian [15]. It is relevant for us, because the leading term in the $6D$ version in this Lagrangian coincides with the Lagrangian of $\mathcal{N} = (1, 1)$ $6D$ SYM theory, the point of our primary interest here. But we consider first more simple examples carrying all the salient features of the complicated field-theoretical and string models.

4.1 Maximal SQM

We consider first the so-called *maximal* $N = 16$ SQM obtained by dimensional reduction from $N = 4$, $4D$ SYM theory or $N = 1$, $10D$ SYM theory. It is convenient to describe the maximal SQM in $10D$ notations and write

$$L = \frac{1}{2}\dot{A}_I^A \dot{A}_I^A - \frac{g^2}{4}f^{ABE}f^{CDE}A_I^A A_J^B A_I^C A_J^D + \frac{i}{2}\lambda_\alpha^A \dot{\lambda}_\alpha^A - \frac{ig}{2}f^{ABC}\lambda_\alpha^A (\Gamma_I)_{\alpha\beta}\lambda_\beta^B A_I^C, \quad (4.1)$$

$I = 1, \dots, 9$, λ_α^A are real fermions, $\alpha = 1, \dots, 16$, and $(\Gamma_I)_{\alpha\beta} = (\Gamma_I)_{\beta\alpha}$ are the $10D$ gamma matrices, $\Gamma_I \Gamma_J + \Gamma_J \Gamma_I = 2\delta_{IJ}$.

¹¹ \approx means “equals modulo a total derivative”.

We consider the simplest such model with $SU(2)$ gauge group. As is well known, the non-Abelian field strength $\hat{F}_{IJ} = [\hat{A}_I, \hat{A}_J]$ and the quartic classical potential $V \sim \langle (\hat{F}_{IJ})^2 \rangle$ vanish in the *Abelian valley*,

$$A_I^A = A_I c^A, \quad (4.2)$$

It is also known that the bottom of this valley is not lifted by quantum corrections so that the system tends to spread along the valley. In the region $g|\mathbf{A}|^3 \gg 1$, one can evaluate the effective Lagrangian depending only on the slow variables A_I and its superpartners in the Born–Oppenheimer framework as a series over the small BO parameter,

$$\gamma = \frac{1}{g|\mathbf{A}|^3}. \quad (4.3)$$

To the lowest order in γ , the Lagrangian is just

$$L_0 = \frac{1}{2}\dot{A}_I^2 + \frac{i}{2}\lambda_\alpha \dot{\lambda}_\alpha. \quad (4.4)$$

It is invariant under supersymmetry transformations

$$\delta_0 A_I = -i\epsilon \Gamma_I \lambda, \quad \delta_0 \lambda = \dot{A}_I \Gamma_I \epsilon, \quad (4.5)$$

It is not so difficult to prove that the action based on the free Lagrangian (4.4) is the *only* one invariant under (4.5) [16]. The effective BO Lagrangian involves, however, many other nontrivial terms. As was mentioned, no potential is generated. Also, there are no corrections to the metric (quadratic in derivative terms). The first relevant correction is quartic in derivatives,

$$L_1 \propto \frac{(\mathbf{E}^2)^2}{g^3|\mathbf{A}|^7} + \text{fermion terms}, \quad (4.6)$$

($\mathbf{E} = \dot{\mathbf{A}}$). As was promised in the Introduction, it is suppressed as γ^3 , compared to (4.4).¹²

The bosonic part (4.6) of L_1 can be determined by evaluating of the graph in Fig. 2. Using the background field formalism, one derives

$$L_1 \propto g^4 (\mathbf{E}^2)^2 \int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 + g^2 \mathbf{A}^2)^4}, \quad (4.7)$$

which gives (4.6). But to restore all other terms involving λ_α is a rather nontrivial task. The full expression was derived in [17]. We will not reproduce it here.

The contribution S_1 in the effective BO action is not invariant under (4.5). On the other hand, the whole series (1.2) represents the effective BO Lagrangian for the original non-Abelian model (4.1). The latter has the full off-shell $\mathcal{N} = 16$ supersymmetry. The effective Lagrangian also must have it. And it has, only the supersymmetry transformations are now *modified* by the same token as they were in the simplest SQM example considered in the previous section. The modified transformations represent an infinite series (1.3) with certain δ_1, δ_2 etc. A (rather complicated) expression for δ_1 can be found in [17].

¹²It is best seen in the Hamiltonian language, where the characteristic values of the canonical momentum \mathbf{E} for the low-lying levels are $E_{\text{char}} \sim A_{\text{char}}^{-1}$.

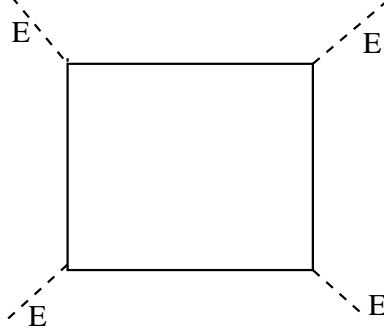


Figure 2: The graph giving the effective Lagrangian (4.7). Solid lines are the propagators of the “fast” charged modes, A_I^\pm .

Expanding $\delta L \approx 0$ in γ and keeping the terms $\sim \gamma^3$, we derive the relation (3.16). This means that, even though the Lagrangian L_1 is not invariant under δ_0 off shell, it *is* invariant on shell — with taking the equations of motion of the Lagrangian (4.4) into account.

It is instructive to check it explicitly. The equations of motion have in this case a very simple form,

$$\ddot{\mathbf{A}} \equiv \dot{\mathbf{E}} = 0, \quad \dot{\lambda}_\alpha = 0. \quad (4.8)$$

If one neglects in L_1 the terms that vanish on mass shell [vanish under the conditions (4.8)], the expression for L_1 greatly simplifies. In [18], it was expressed as

$$L_1^{\text{mass shell}} \propto \frac{(\mathbf{E}^2)^2}{|A_I - i(E_J/2\mathbf{E}^2) \lambda \Gamma^{IJ} \lambda|^7}, \quad (4.9)$$

where $\Gamma^{IJ} = \Gamma^{[I} \Gamma^{J]}$. The variation of (4.9) under (4.5) amounts to a total time derivative plus the terms involving $\ddot{\mathbf{A}}$ and $\dot{\lambda}_\alpha$. Indeed, it is easy to see that, under the conditions (4.8),

$$\delta_0 \left(A_I - i \frac{E_J}{2\mathbf{E}^2} \lambda \Gamma^{IJ} \lambda \right) = \frac{i E_I E_K}{\mathbf{E}^2} \lambda \Gamma^{K\epsilon} \epsilon. \quad (4.10)$$

Then

$$\delta_0 L_1^{\text{mass shell}} = f(\mathbf{E}, \lambda_\alpha) E^I \frac{\partial}{\partial A_I} g(\mathbf{A}, \mathbf{E}, \lambda)$$

with some f, g . Using again (4.8), this gives

$$\delta_0 L_1^{\text{mass shell}} \stackrel{\text{on mass shell}}{\equiv} f \frac{d}{dt} g \stackrel{\text{on mass shell}}{\equiv} \frac{d}{dt} (fg). \quad (4.11)$$

The bosonic effective Lagrangian involves also the terms with still higher derivatives. For example, there is a term

$$L_2 \propto \frac{(\mathbf{E}^2)^3}{|\mathbf{A}|^{14}} \quad (4.12)$$

that is suppressed as γ^6 , compared to (4.4). The coefficient in (4.12) was evaluated in an accurate 2-loop calculation in [19]. The full expression for L_2 including fermions is not

known. Neither is known the simplified expression for L_2 disregarding the terms proportional to (4.8).

Note that this expression *does not* need to be invariant with respect to (4.5) on mass shell and is probably not. Indeed, keeping the terms of order $\sim \gamma^6$, we can only derive the relation (3.19). The last term there is proportional to the equations of motion (4.8). But the term $\delta_1 L_1$ is not, and one cannot make any conclusions about $\delta_0 L_2$. Probably, the true L_2 , entering the true BO effective Lagrangian, is not on-shell supersymmetric, even though a supersymmetric on-shell invariant can in principle be written:

$$\tilde{L}_2 \propto \frac{(\mathbf{E}^2)^3}{|A_I - i(E_J/2\mathbf{E}^2) \lambda \Gamma^{IJ} \lambda|^{14}}. \quad (4.13)$$

4.2 Tree chiral Lagrangian

We now go back to the effective chiral theory, being not concerned this time with loop corrections, but only with the structure of the leading term (2.1) and of the tree amplitudes that it generates.¹³

Expanding the exponentials, we can present the Lagrangian (2.1) as an infinite series,

$$\mathcal{L}^{(0)} = \mathcal{L}_0 + \mathcal{L}_1 + \dots,$$

the beginning of this series being written in (2.3). The full Lagrangian is invariant under the transformations,

$$U \rightarrow \exp\{i\sigma^a \alpha^a\} U, \quad U \rightarrow U \exp\{i\sigma^a \beta^a\}. \quad (4.14)$$

Consider e.g. the left multiplication. Infinitesimally, it gives

$$\delta_\alpha \phi^a = -\varepsilon^{abc} \alpha^b \phi^c + \frac{|\phi|}{\text{tg}|\phi|} \alpha^a + \frac{\alpha \cdot \phi}{|\phi|^2} \left[1 - \frac{|\phi|}{\text{tg}|\phi|} \right] \phi^a, \quad (4.15)$$

where $\phi^a = \pi^a/F_\pi$. If also ϕ is small, this can be presented as

$$\delta_\alpha \phi^a = -\varepsilon^{abc} \alpha^b \phi^c + \alpha^a + \frac{1}{3} \alpha^b (\phi^b \phi^a - \delta^{ab} \phi^2) + O(\phi^4) \equiv -\varepsilon^{abc} \alpha^b \phi^c + \delta_0 \phi + \delta_1 \phi + \dots \quad (4.16)$$

The first term in (4.16) describes the $SO(3)$ rotations, the diagonal symmetry $U \rightarrow \Omega U \Omega^\dagger$, with respect to which all terms of the expansion (2.3) are still invariant. But it is not true for other contributions in (4.16). Actually, we may now observe that the leading term in the expansion is still invariant under the translations $\delta_0 \phi^a = \alpha^a$. In addition, we may observe that the variation $\delta_0 \mathcal{L}_1$ amounts to a total derivative, if taking into account the tree equations of motion $\square \phi^a = 0$. In other words, $\delta_0 \mathcal{L}_1$ vanishes on mass shell! In the full analogy with the maximal SQM model and with the SQM model of Sect. 3, this is a direct corollary of the fact that the full series in (2.3) is invariant under the full series in (4.16).

Note that the term $\sim \pi^6$ in the expansion (we need not to write it explicitly) is *not* invariant on shell under translations, it only satisfies the condition (3.19). This means in particular that the tree 6-point amplitude generated by \mathcal{L}_2 does not exhibit the full chiral symmetry $SU_L(2) \times SU_R(2)$, it only keeps its diagonal $SO(3)$ part. On the other hand, the *full* tree amplitude, two relevant contributions to which being depicted in Fig. 3, is of course $SU_L(2) \times SU_R(2)$ - symmetric.

¹³I am indebted to G. Bossard who attracted my attention to this example.

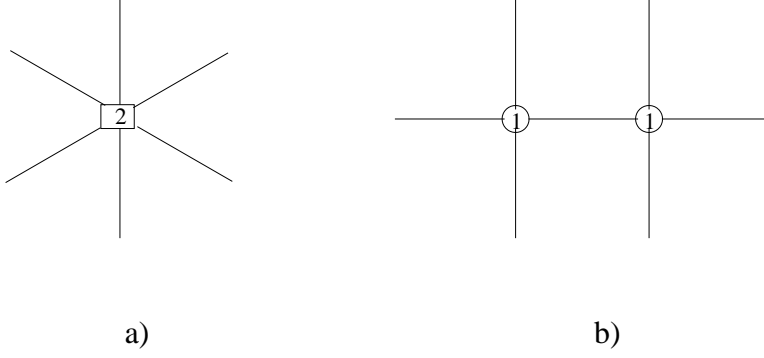


Figure 3: Contributions of \mathcal{L}_1 and \mathcal{L}_2 to 6-point amplitudes.

5 Six-dimensional gauge theories

In $5+1$ Minkowski space, left-handed spinors λ^a and right-handed spinors ψ_a ($a = 1, 2, 3, 4$) belong to the different completely independent spinor representations $(1,0)$ and $(0,1)$ of $Spin(5,1)$.¹⁴ We introduce six 4×4 matrices γ_M (the $6D$ analogs of σ_μ), which are real, antisymmetric and satisfy

$$\gamma_M \tilde{\gamma}_N + \gamma_N \tilde{\gamma}_M = -2\eta_{MN}, \quad \eta_{MN} = \text{diag}(1, -1, -1, -1, -1, -1), \quad (5.1)$$

with

$$(\tilde{\gamma}_M)^{ab} = \frac{1}{2} \varepsilon^{abcd} (\gamma_M)_{cd}. \quad (5.2)$$

The minimal $\mathcal{N} = (1,0)$ supersymmetric Yang-Mills theory in 6 dimensions includes the gauge potential A_M and a couple of left-handed fermion fields λ_j^a ($j = 1, 2$) satisfying the pseudoreality condition,

$$\overline{\lambda_j^a} \equiv -C^a_b (\lambda_j^b)^* = \lambda^{aj} \equiv \varepsilon^{jk} \lambda_k^a, \quad (5.3)$$

where C is the charge conjugation matrix with the properties $C = -C^T$, $C^2 = -1$. In addition, the gauge multiplet involves the triplet of auxiliary fields $\mathcal{D}_{jk} = \mathcal{D}_{kj}$. All the fields represent Hermitian colour matrices.

Being expressed in components, the Lagrangian reads

$$\mathcal{L}^{\text{gauge}} = \frac{1}{2f^2} \langle -F_{MN}^2 + i\lambda^k \gamma^M \nabla_M \lambda_k - \mathcal{D}_{jk} \mathcal{D}^{jk} \rangle, \quad (5.4)$$

with $\nabla_M X = \partial_M X - i[A_M, X]$, $F_{MN} = [\nabla_M, \nabla_N]$. The constant f^2 carries the dimension m^{-2} , the same as Newton's gravity constant and as F_π^{-2} .

As this theory includes only left-handed fields, it involves a chiral anomaly [20], which breaks gauge invariance. To compensate this anomaly, one should add an adjoint matter

¹⁴In Euclidean space, two spinor representations of $Spin(6) = SU(4)$ are complex conjugate to one another. The situation is exactly opposite to that in four dimensions, where two spinor representations are conjugate to one another in Minkowski space, but are completely independent in Euclidean space.

hypermultiplet, involving the right-handed pseudoreal fermions $\psi_a^{A=1,2}$ and four real scalars φ_α . The corresponding Lagrangian reads

$$\mathcal{L}^{\text{hyper}} = \frac{1}{2f^2} \langle -i\psi^A \tilde{\gamma}^M \nabla_M \psi_A + (\nabla_M \varphi_\alpha)^2 - \psi_a^A [\lambda^{ka}, \varphi_{kA}] - \varphi^{kA} \varphi_A^l \mathcal{D}_{kl} \rangle \quad (5.5)$$

with ¹⁵

$$\varphi^{kA} = \frac{1}{\sqrt{2}} (\sigma_\alpha)^{kA} \varphi_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} i\varphi_0 + \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & i\varphi_0 - \varphi_3 \end{pmatrix}. \quad (5.6)$$

To provide for the extended supersymmetry of the sum $\mathcal{L}^{(1,1)} = \mathcal{L}^{\text{gauge}} + \mathcal{L}^{\text{hyper}}$, the constant f in (5.5) is chosen to be the same as in (5.4). After excluding the auxiliary fields \mathcal{D}_{jk} , we obtain

$$\begin{aligned} \mathcal{L}^{(1,1)} = \frac{1}{2f^2} \langle & -F_{MN}^2 + i\lambda^k \gamma^M \nabla_M \lambda_k - i\psi^A \tilde{\gamma}^M \nabla_M \psi_A + (\nabla_M \varphi_\alpha)^2 \\ & - \psi_a^A [\lambda^{ka}, \varphi_{kA}] + \frac{1}{16} [\varphi^{kA}, \varphi_A^l] [\varphi_k^B, \varphi_{lB}] \rangle \end{aligned} \quad (5.7)$$

One can be convinced that the corresponding action is invariant, indeed, under certain nonlinear $\mathcal{N} = (1, 1)$ supersymmetry transformations with left-handed and right-handed Grassmann parameters.

In contrast to Witten's models of Sect. 2, to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ 4D SYM theories, but similar to the maximal SQM model of Sect. 3 and to $\mathcal{N} = 4$ 4D SYM theory, a linear superfield realization of this supersymmetry is not known and probably does not exist.

As a result, one cannot write an off-shell $\mathcal{N} = (1, 1)$ invariant of a fixed canonical dimension $d > 4$. On the other hand, higher-dimensional structures that possess $\mathcal{N} = (1, 1)$ supersymmetry *on shell* [i.e. the structures invariant under the $\mathcal{N} = (1, 1)$ supertransformations when the fields are constrained to satisfy the equations of motion of the basic Lagrangian (5.7)] do exist. Such a restricted on-shell invariance is *good enough* for these structures to play the role of counterterms for the theory (5.7) and to give rise to logarithmic divergences in its on-mass-shell scattering amplitudes. ¹⁶

5.1 Harmonic superspace and harmonic superfields

There is no off-shell $\mathcal{N} = (1, 1)$ superfield formalism, but the $\mathcal{N} = (1, 0)$ off-shell superfields well exist, can be studied and can be used. Explicit expressions for superfields realizing irreducible representations of the supersymmetry algebra can be best written in the framework of the harmonic superspace approach [14]. We address the reader to our recent paper [3] for a detailed description of this formalism, as applied to 6-dimensional theories, and quote here only its salient features.

The standard $\mathcal{N}=(1, 0)$ superspace involves the coordinates

$$z = (x^M, \theta^{ai}), \quad (5.8)$$

¹⁵Note the constraint $\varphi^{kA} = -(\varphi_{kA})^*$ that the fields (5.6) satisfy.

¹⁶One can also mention here the existence of nontrivial higher-derivative actions enjoying on-shell (but not off-shell) $\mathcal{N} = 4$ supersymmetry in four dimensions (see, e.g., [21, 22]). But these invariants are not relevant in perturbative calculations, they do not appear as counterterms for the renormalizable, and even finite $\mathcal{N} = 4$ 4D theory.

where θ^{ai} are Grassmann pseudoreal left-handed spinors.

Next we introduce the harmonics $u^{\pm i}$ [$u_i^- = (u_i^+)^*$, $u^{+i}u_i^- = 1$], which describe the “harmonic sphere” $SU(2)_R/U(1)$, where $SU(2)_R$ is R-symmetry group of the $\mathcal{N} = (1, 0)$ Poincaré superalgebra. We now consider the projections $\theta^{\pm a} = u_k^{\pm} \theta^{ak}$ and introduce the “analytic coordinate” $x_{(\text{an})}^M = x^M + \frac{i}{2} \theta^+ \gamma^M \theta^-$.

A very important property is that the set of coordinates

$$\zeta := (x_{(\text{an})}^M, \theta^{+a}, u^{\pm i}), \quad (5.9)$$

involving only a half of the original Grassmann coordinates forms a subspace closed under the action of $\mathcal{N}=(1, 0)$ 6D supersymmetry. The set (5.9) parametrizes what is called “harmonic analytic superspace”.

Many relevant superfields are *Grassmann-analytic* or *G-analytic*, which means that they do not depend in the analytic basis (5.9) on θ^- , but only on θ^+ . This makes a series over θ much shorter and much more handleable than for a generic superfield. G-analytic superfields are quite analogous to habitual chiral superfields in ordinary 4D $\mathcal{N} = 1$ superspace, which depend either only on θ or only on $\bar{\theta}$.

It is convenient to define the differential operators D^+ and $D^{\pm\pm}$ called spinor and harmonic derivatives. In the analytic basis, they are expressed as

$$D_a^+ = \frac{\partial}{\partial \theta^{-a}}, \quad D^{\pm\pm} = u^{\pm i} \frac{\partial}{\partial u^{\mp i}} + \frac{i}{2} \theta^{\pm a} \gamma^M \theta^{\pm b} \frac{\partial}{\partial x^M} + \theta^{\pm a} \frac{\partial}{\partial \theta^{\mp a}}. \quad (5.10)$$

We also define the operator of harmonic charge,

$$D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{+a} \frac{\partial}{\partial \theta^{-a}} - \theta^{-a} \frac{\partial}{\partial \theta^{+a}} \quad (5.11)$$

and classify the superfields by the eigenvalues of D^0 . The superfield with eigenvalue 1 will be denoted as X^+ , the superfield with eigenvalue -2 as Y^{--} , etc.

A supersymmetric action can be obtained by integrating a generic superfield of zero harmonic charge over the whole superspace or by integrating an analytic superfield of harmonic charge $+4$ over the analytic superspace. The corresponding measures will be denoted

$$dZ = d^6x d^8\theta du, \quad d\zeta^{(-4)} = d^6x_{\text{an}} d^4\theta^+ du, \quad (5.12)$$

where du is the measure on the harmonic sphere (with the normalization $\int du = 1$), $d^8\theta = d^4\theta^+ d^4\theta^-$ and we have chosen the convention $\int d^4\theta^+ (\theta^{+a} \theta^{+b} \theta^{+c} \theta^{+d}) = -\varepsilon^{abcd}$.

The gauge supermultiplet is described by a G-analytic superfield V^{++} . Its component expansion in the Wess-Zumino gauge is very simple,

$$V^{++} = \frac{1}{2} \theta^+ \gamma^M \theta^+ A_M - \frac{1}{3} \varepsilon_{abcd} \theta^{+b} \theta^{+c} \theta^{+d} \lambda^{-a} + \frac{1}{8} \varepsilon_{abcd} \theta^{+a} \theta^{+b} \theta^{+c} \theta^{+d} \mathcal{D}^{--}, \quad (5.13)$$

where $\mathcal{D}^{--} = \mathcal{D}^{jk} u_j^- u_k^-$.

Given the superfield V^{++} , one can also define the superfield V^{--} from the requirement that the commutator $[\nabla^{++}, \nabla^{--}]$ of the covariant harmonic derivatives,

$$\nabla^{++} = D^{++} + V^{++}, \quad \nabla^{--} = D^{--} + V^{--}, \quad (5.14)$$

is the same as $[D^{++}, D^{--}] = D^0$. In contrast to the G-analytic V^{++} , V^{--} is a generic superfield. In the following, we will also need the superfields

$$W^{+a} = -\frac{1}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+V^{--}, \quad F^{++} = \frac{1}{4}D_a^+W^{+a} = -\frac{1}{24}\varepsilon^{abcd}D_a^+D_b^+D_c^+D_d^+V^{--}. \quad (5.15)$$

The superfield F^{++} is G-analytic.

The minimal $\mathcal{N} = (1, 0)$ SYM action is described via V^{++} as follows [23],

$$S^{SYM} = \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left\langle \int d^6x d^8\theta du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)} \right\rangle, \quad (5.16)$$

Substituting there (5.13), we can reproduce (5.4). Note that the superfield equations of motion for the action (5.16) are extremely simple,

$$F^{++} = 0. \quad (5.17)$$

To describe the supermultiplet interactions, we have to introduce the pseudoreal G-analytic superfield q^{+A} . The minimal $d = 4$ Lagrangian (5.5) follows from the superfield action

$$S = -\frac{1}{2f^2} \left\langle \int d\zeta^{-4} q^{+A} \nabla^{++} q_A^+ \right\rangle. \quad (5.18)$$

In the free case, $\nabla^{++} \rightarrow D^{++}$ and the field q^{+A} satisfies the equations of motion $D^{++}q^{+A} = 0$. They can be resolved to obtain

$$q^{+A} = \varphi^{+A} - \theta^{+a}\psi_a^A - \frac{i}{2}\theta^{+a}\gamma^M\theta^{+b}\partial_M\varphi^{-A}, \quad (5.19)$$

where $\varphi^{\pm A} = \varphi^{kA}u_k^{\pm}$, with φ^{kA} being the physical harmonic-independent pseudoreal [see the footnote to Eq.(5.6)] *on-shell* scalar fields satisfying the d'Alembert equation $\square\varphi = 0$, and ψ_a^A are the physical right-handed on-shell pseudoreal fermionic fields satisfying the free Dirac equation.

Generically, q^{+A} involves an infinite number of other component fields. It turns out, however, that, when one is only interested in the minimal $d = 4$ hypermultiplet action, they all enter the Lagrangian without derivatives and can be algebraically excluded.

The Lagrangian (5.5) is obtained by substituting (5.19), (5.14) and (5.13) in (5.18).

One can be convinced that the sum of the gauge action (5.16) and the hypermultiplet action (5.18) is invariant under the following $\mathcal{N} = (0, 1)$ supersymmetry transformations with right-handed pseudoreal Grassmann parameters ϵ_{aA} ,

$$\begin{aligned} \delta_0 V^{++} &= \epsilon^{+A} q_A^+, \\ \delta_0 q_A^+ &= -(D^+)^4 (\epsilon_A^- V^{--}) \end{aligned} \quad (5.20)$$

($\epsilon_A^{\pm} = \epsilon_{aA}\theta^{\pm a}$). The $\mathcal{N} = (1, 0)$ symmetry is of course manifest.

6 Effective Lagrangian and counterterms

As was mentioned in Sect. 2, in order to perform practical calculations in effective chiral theory, one should include in the bare Lagrangian an infinite number of counterterms with UV-divergent coefficients. These divergences cancel order by order, while calculating the loops. However, the $6D$ theory described in the previous section does not have practical experimental applications. We do not really want to calculate the amplitudes, but only wish to elucidate the structure of UV divergences in this theory. It is then convenient for us to assume that the bare Lagrangian is just (5.7). The effective Wilsonean Lagrangian has then an explicit dependence on the ultraviolet cutoff Λ . We will be interested in the higher-derivative contributions in this effective Lagrangian. Having not found another good word, we will still call them "counterterms".

It is more or less clear that, when presented in such a way, the full Wilsonean effective Lagrangian for this theory should possess the same symmetry, the full $\mathcal{N} = (1, 1)$ supersymmetry, as the tree Lagrangian. Let us justify this important claim at the *physical* level of rigour.

- We note first that the Wilsonean effective Lagrangian has the same nature as any other effective Lagrangian — it is obtained by integrating out high-momenta and high-energy modes.
- Consider the corresponding effective Hamiltonian. Its spectrum should match the low-energy spectrum of the original Hamiltonian and enjoy, in particular, the same degeneracies.
- But if the symmetry is preserved at the Hamiltonian level, this should also be the case for the Lagrangian.

In the SQM examples considered before, we have seen, however, that one cannot keep the full symmetry for the contributions to the effective Lagrangian of a given canonical dimension, if there is no linear (superfield) realization of the full supersymmetry or if this realization is not implemented. Thus, one cannot expect the higher-dimensional counterterms that are relevant for calculations of on-shell scattering amplitudes in the theory (5.7) to have the full $\mathcal{N} = (1, 1)$ supersymmetry.

However, the counterterms should be gauge-invariant. In addition, they should enjoy the off-shell $\mathcal{N} = (1, 0)$ supersymmetry. These constraints follow from the fact that one-particle-irreducible amplitudes calculated at a given loop order should satisfy the Ward identities following from the gauge invariance and $\mathcal{N} = (1, 0)$ supersymmetry.

The first statement (about gauge invariance) is rather common and does not require special comments.¹⁷ The requirement that $\mathcal{N} = (1, 0)$ supersymmetry is preserved follows from the existence of $\mathcal{N} = (1, 0)$ superspace and superfield description, where supersymmetry is realized linearly. In such cases, one can develop a supergraph technique that keeps the $\mathcal{N} = (1, 0)$ supersymmetry by construction. Such a technique has not been explicitly formulated yet, but it should be similar in spirit to the $\mathcal{N} = 2$ $4D$ supergraph technique described in [14].

Anyway, we assume that explicit perturbative calculations in [2] manifestly keep this symmetry at each loop order, even though they are not supergraph calculations. This claim is confirmed by the *results* obtained there.

¹⁷It is important, of course, that the theory (5.7) is anomaly-free.

We are now going to show that the relevant counterterms do not arise at the 1-loop and 2-loop level.

6.1 $d = 6$

Consider first possible 1-loop counterterms. Their canonical dimension should be $d = 6$. The only gauge-invariant $\mathcal{N} = (1, 0)$ supersymmetric $d = 6$ action involving the gauge supermultiplet V^{++} is [24]

$$S_{d=6}^{\text{gauge}} \sim \left\langle \int d\zeta^{-4} (F^{++})^2 \right\rangle, \quad (6.1)$$

with the G-analytic superfield F^{++} defined in (5.15). However, the equations of motion for the pure gauge theory (5.16) are exactly $F^{++} = 0$, i.e. the action (6.1) vanishes on mass shell and is irrelevant.

If we include the hypermultiplet, the equations of motion are modified to

$$F^{++} + \frac{1}{2}[q^{+A}, q_A^+] = 0. \quad (6.2)$$

The pure gauge action (6.1) does not vanish on mass shell anymore, A generic $d = 6$ action represents [25] a linear combination of (6.1), of the structure

$$S_{\text{quart}} \sim \left\langle \int d\zeta^{-4} [q^{+A}, q_A^+]^2 \right\rangle \quad (6.3)$$

and of an infinite series of structures

$$S_n \sim \left\langle \int dZ q^{+A} (\nabla^{--})^n (\nabla^{++})^{n-1} q_A^+ \right\rangle. \quad (6.4)$$

A generic linear combination does not vanish on the mass shell (6.2). We have seen, however, that the full extended supersymmetry of the full effective Lagrangian $L_0 + L_1 + \dots$ implies that the tree-level supersymmetry variation $\delta_0 L_1$ of the next-to-leading term vanishes on mass shell [see the discussion after Eq.(3.16)]. In our case, L_0 is the tree action, δ_0 are the transformations (5.20) and we are studying the question if L_1 might have canonical dimension $d = 6$. One can show [3] that the requirement for $\delta_0 L^{d=6}$ to vanish on mass shell leads to the conclusion that $L^{d=6}$ vanishes on mass shell itself. Which means that logarithmic divergences are absent at the 1-loop level.

6.2 $d = 8$

We go over to two loops. Consider first possible $\mathcal{N} = (1, 0)$ off-shell supersymmetric counterterms of canonical dimension $d = 8$ in the pure gauge sector. One can show that all of them vanish on mass shell.¹⁸

The analysis of the structures including the hypermultiplet was performed in Ref.[3]. At the first step, one can show that all the possible structures can be reduced on shell to

$$S_{\text{quart}}^{d=8} \sim \left\langle \int dZ [q^{-A}, q_A^-] [q^{+B}, q_B^+] \right\rangle, \quad (6.5)$$

¹⁸This has been known since [26], but we address the reader to [3] for a simple “harmonic” proof.

where $q^{-A} = \nabla^{--} q^{+A}$.

At the second step, one can calculate the variation of (6.5) under the transformations (5.20) and find out that this variation does not vanish on shell. Hence, the corresponding Lagrangian does not satisfy the requirement (3.16) and cannot represent a next-to-leading term \mathcal{L}_1 in the Wilsonian effective Lagrangian. Note that we used in this reasoning the already established fact that there are no $d = 6$ counterterms and the first correction (if any) should start from $d = 8$.

On the other hand, if we lift the requirement of off-shell $\mathcal{N} = (1, 0)$ invariance, nontrivial $d = 8$ invariants possessing on-shell $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ symmetries can be written. Again, consider first the pure gauge sector. We can write [27, 3]

$$S_1^{d=8} \sim \left\langle \int d\zeta^{-4} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d} \right\rangle. \quad (6.6)$$

This action is not $\mathcal{N} = (1, 0)$ invariant, because W^{+a} is not a G-analytic superfield, $D_a^+ W^{+a} = 4F^{++} \neq 0$, and the integrand in (6.6) is not a G-analytic superfield. But (6.6) is $\mathcal{N} = (1, 0)$ invariant *on shell*. Indeed, in the pure gauge case, F^{++} vanishes on shell and hence W^{+a} is G-analytic on shell!

Being expressed through components, the bosonic part of (6.6) gives the known F^4 structure [28] (see [29] for the complete component form),

$$[\mathcal{L}_1^{d=8}]_{\text{bos}} \sim \left\langle 2F_{MN} F^{MN} F_{PQ} F^{PQ} + F_{MN} F_{PQ} F^{MN} F^{PQ} - 4F^{NM} F_{MR} F^{RS} F_{SN} - 8F^{NM} F_{MQ} F_{NR} F^{RQ} \right\rangle_{(s)}. \quad (6.7)$$

Note the presence of the symmetrized color trace $\langle \rangle_{(s)} \sim \langle X(YZU + UYZ + ZUY) \rangle$ in (6.7).

The invariant (6.6) involves a single colour trace. One can also write a double-trace invariant,

$$S_2^{d=8} \sim \int d\zeta^{-4} \varepsilon_{abcd} \langle W^{+a} W^{+b} \rangle \langle W^{+c} W^{+d} \rangle. \quad (6.8)$$

By including the hypermultiplet, both (6.6) and (6.8) can be completed to the $\mathcal{N} = (1, 1)$ on-shell invariant actions. The explicit expressions can be found in Ref.[3].

What is the physical relevance of these new invariants? As was explained earlier, they are not admissible counterterms in the Wilsonian Lagrangian for the theory (5.7) — there are no UV divergences at the 2-loop level. However, the single-trace invariant *does* appear in the effective field theory actions for certain string theories and, in particular, in the derivative expansion of the Born-Infeld effective action for coincident $D5$ -branes [15].

6.3 $d = 10$ and beyond

At the 3-loop level, logarithmic ultraviolet divergences and the associated counterterms do appear. In the pure gauge theory, one can write down two different structures,

$$S_1^{d=10} \sim \left\langle \int dZ \varepsilon_{abcd} W^{+a} W^{-b} W^{+c} W^{-d} \right\rangle \quad (6.9)$$

and

$$S_2^{d=10} \sim \int dZ \varepsilon_{abcd} \langle W^{+a} W^{-b} \rangle \langle W^{+c} W^{-d} \rangle. \quad (6.10)$$

$$(W^{-a} = \nabla^{--} W^{+a}).$$

In contrast to (6.6) and (6.8), the actions (6.9) and (6.10) are invariant under the $\mathcal{N} = (1, 0)$ transformations off shell — the integrands are not G-analytic, but generic superfields of zero harmonic charge, and the integral is done over the whole superspace. Incidentally, the extra θ^- integration brings about the extra two dimensions of mass, compared to (6.6) and (6.8), so that the invariants (6.9) and (6.10) carry dimension 10 rather than 8. Using a special harmonic *on-shell* superfield technique, suggested first in [30], one has succeeded in expressing explicitly the on-shell $\mathcal{N} = (1, 1)$ supersymmetric completions of the actions (6.9) and (6.10) [3].

And here we meet a puzzle. The existence of two different $d = 10$ invariants suggests the presence of two different logarithmically divergent 3-loop contributions to the scattering amplitudes: a single-trace and a double-trace one. The explicit calculations of [2] confirmed the absence of ultraviolet logarithms at the 1-loop and 2-loop levels, but did not confirm the presence of two different logarithmically divergent 3-loop structures — only the single-trace structure was seen. We do not now understand why the single-trace counterterm is selected and the double-trace is not. Hopefully, a more meticulous study combining the harmonic superspace methods with cohomology arguments of Ref.[30] could provide an answer to this question.

As was explained above, the full Wilsonian effective Lagrangian $\mathcal{L} = \mathcal{L}^{(d=4)} + \mathcal{L}^{(d=10)} + \dots$ should be invariant under the modified supersymmetry transformations, $\delta = \delta_0 + \delta_1 + \dots$, where δ_0 was given in (5.20), and δ_1 has the order $\sim (g\partial)^6 \delta_0$, with ∂ meaning an extra spatial derivative. The effective Lagrangian can also include the terms of still higher dimension, $\sim \mathcal{L}^{(d=12)} + \mathcal{L}^{(d=14)} + \dots$. The same concerns the modified transformations. The variations $\delta_0 \mathcal{L}^{(d=12)}$ and $\delta_0 \mathcal{L}^{(d=14)}$ should still vanish on mass shell. This follows from the condition (3.16), where one can include $\mathcal{L}^{(d=12)}$ and $\mathcal{L}^{(d=14)}$ into \mathcal{L}_1 and the corresponding higher-derivative terms in the supersymmetry transformations into δ_1 . The situation becomes more complicated at the level $d = 16$. $\mathcal{L}^{(d=16)}$ satisfies a more complicated condition (3.19) with nonzero $\delta_1 \mathcal{L}_1 \equiv \delta_1 \mathcal{L}^{(d=10)}$ and need not be on-shell supersymmetric. But the amplitudes *are* supersymmetric - cf. the discussion in Sect. 4.2.

7 Lessons for supergravity

As was mentioned, in Einstein's gravity, the first relevant counterterm (2.11) has dimension $d = 6$ and shows up at the 2-loop level. Since 40 years, it has been known that the structure (2.11) cannot be supersymmetrized: the effective Lagrangian (2.11) generates helicity-flip amplitudes, which is not compatible with $\mathcal{N} = 1$ supersymmetry [31].

At the 3-loop level, we can have the structure $\sim R^4 + \dots$, which is not protected by this argument [32]. That means that logarithmic UV divergences may appear in the $\mathcal{N} = 1$ supergravity at the 3-loop level (though, to the best of our knowledge, this has not yet been directly checked).

Extended supersymmetries and, especially, the maximal $\mathcal{N} = 8$ supersymmetry bring about further constraints. To establish them is a more difficult task than for the extended $6D$ SYM theory discussed in Sect. 5.6. First, because the dimension of the “dangerous” structures that one should study for supergravity is higher and the structures are more complicated. Second, because of a much wider “gap” between the superfield description (which is possible only for the minimal $\mathcal{N} = 1$ supersymmetry) and the extended $\mathcal{N} = 8$

on-shell supersymmetry, the presence or the absence of which for the candidate counterterms one has to establish.

Since [33], people have been aware of the presence of the counterterm of dimension $d = 18$ that satisfies all the on-shell symmetries in the $\mathcal{N} = 8$ theory. It should bring about logarithmic UV divergences at eight loops. At that time, it was not clear, however, whether also some lower-dimensional counterterms (starting from the 3-loop level) are or are not admissible.

The interest to this problem was resuscitated in the new century, owing to the works of Bern’s group that displayed the absence of logarithmic UV divergences at the 3-loop and then at the 4-loop level [1]. This fact needed to be explained. After several years of hard work (see e.g. [22, 30, 34]), people understood that the extended $\mathcal{N} = 8$ supersymmetry and other related on-shell symmetries exclude the presence of counterterms up to $d = 14$. This means that on-shell scattering amplitudes should be free from logarithmic UV divergences through 6 loops.

The frontier of unknown is now pushed up to seven loops. An invariant of dimension $d = 16$, $\mathcal{L} \sim \partial^8 R^4 + \dots$ which seems to satisfy all the on-shell symmetry requirements was constructed [35]. It suggests the presence of UV logarithms at 7 loops.

We cannot now be sure, however, that these divergences are indeed there, because there are at least three occasions where known theoretical arguments failed to justify certain noteworthy cancellations seen in “experiment” — in explicit perturbative calculations. The first such example was discussed in Sect. 6 — the calculations displayed the absence of the divergences associated with the double-trace $d = 10$ structure in 3-loop calculations in $6D$ SYM theory. The second and the third examples are the extended, but not maximally extended $4D$ supergravities with $\mathcal{N} = 4$ and $\mathcal{N} = 5$. For $\mathcal{N} = 4$, one can build up a 3-loop on-shell invariant [32] and, for $\mathcal{N} = 5$ a 4-loop on-shell invariant [34, 35]. But the explicit calculations [36] displayed the absence of the divergences at this level. Another known example of unexpected cancellations refers to a certain gauged supergravity where the beta function was shown to vanish at the 1-loop level [37].

Maybe 7-loop divergences in the $\mathcal{N} = 8$ supergravity are also killed by an unknown yet reason, in which case the logarithmic UV divergences only appear starting from eight loops, as was anticipated back in 1980 [33]...

Strictly speaking, and especially bearing in mind the just mentioned mismatch, indicating the absence of full understanding, we cannot be quite sure of that until explicit 7-loop and 8-loop calculations in $\mathcal{N} = 8$ supergravity are done. Unfortunately, there is a little hope to see such calculations in the foreseeable future. Our bet, however, is that the logarithmic divergences appear at some level, counterterms start playing a role and there is an infinite number of them. This means that $\mathcal{N} = 8$ supergravity has essentially the same ultraviolet behaviour as Fermi theory and as chiral theory and cannot be considered as fundamental.

The question of whether higher-dimensional counterterms are relevant or not is important. Indeed, it is clear from the discussion in Sect. 2 that non-renormalizability represents a problem not so much due to logarithmic or power UV divergences (they can in principle be removed order by order from physical observables), but due to an uncontrollable energy growth of amplitudes and cross-sections and impossibility to make perturbative calculations beyond the unitarity barrier. Such calculations are definitely impossible in the presence of higher-dimensional counterterms: high-energy amplitudes and high-energy cross sections would essentially depend in this case on an infinite number of dimensionful coupling constants.

Were such extra constants absent, one could contemplate a scenario where the theory would be defined perturbatively at all energies. Suppose (we know that it is wrong, but suppose) that Fermi theory would not involve extra counterterms bringing about logarithmic UV divergences. In this case, everything depended on the physical renormalized Fermi's constant G_F^{phys} . Consider the total cross section of νe scattering. At the tree level, it grows with energy as $\sigma_{\nu e} \sim G_F^2 s$. With loop corrections taken into account and renormalization performed, it would acquire the form

$$\sigma_{\nu e} = G_F^2 s (a_0 + a_1 G_F s + \dots) . \quad (7.1)$$

One could then speculate that the series in the R.H.S. of (7.1) might converge to a function $f(G_F s)$ that falld down with s , so that the global cross section did not grow as a power and the Froissart theorem were fulfilled.

One can ask whether something similar could happen in gravity, at least for first six loops where there are no counterterms. Unfortunately, gravity involves massless particles, this invalidates the Froissart theorem and the whole reasoning above. The *total* graviton-graviton cross section is simply infinite by the same reason as the total Coulomb cross section is infinite — due to the massless propagators in the t -channel. One can calculate differential cross sections allowing for the creation of extra soft gravitons, but it is a complicated object depending on several kinematic invariants including ω_{max} (the maximal allowed total energy for extra soft particles produced) [38]. We do not know how to define it in higher loops, if one wants to keep track not only of the infrared ω_{max} - dependent part, but also of the finite part.

Thus, the Planck mass barrier is difficult (impossible?) to cross in $\mathcal{N} = 8$ supergravity even in a hypothetical improbable case where for some reason all the coefficients of higher-dimensional counterterms vanish...

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References

- [1] *Bern Z., Carrasco J.J., Dixon I.J., Johansson H. and Roiban R.* The Ultraviolet Behavior of $\mathcal{N} = 8$ Supergravity at Four Loops // Phys. Rev. Lett. 2009. V. 103. P. 081301.
- [2] *Bern Z., Carrasco J.J., Dixon I.J., Johansson H. and Roiban R.* The Complete Four-loop Four-point Amplitude in $\mathcal{N}=4$ Super-Yang-Mills Theory // Phys. Rev. D. 2010. V. 82. P. 125040.
- [3] *Bossard G., Ivanov E.A. and Smilga A.V.* Ultraviolet Behavior of 6D Supersymmetric Yang-Mills Theories and Harmonic Superspace // JHEP 2015. V. 1512. P. 085.
- [4] *Witten E.* Dynamical Breaking of Supersymmetry // Nucl. Phys. B. 1981. V. 188. P. 513.

- [5] *Gasser J. and Leutwyler H.* Chiral Perturbation Theory to One Loop // Ann. Phys. 1984. V. 158 P. 142;
Chiral Perturbation Theory: Expansions in the Mass of the Strange Quarks // Nucl. Phys. B. 1985. V. 250. P. 465.
- [6] See e.g. *Shushpanov I.A. and Smilga A.V.* Chiral Perturbation Theory with Lattice Regularization // Phys. Rev. D. 1999. V. 59. 054013.
- [7] *Slavnov A.A.* Invariant Regularization of Nonlinear Chiral Theories // Nucl. Phys. B. 1971. V. 31. P. 301.
- [8] *Froissart M.* Asymptotic Behavior and Subtractions in the Mandelstam Representation // Phys. Rev. 1961 V. 123. P. 1053.
- [9] See e.g. *Makeenko Yu. M., Ter-Martirosian K.A. and Zamolodchikov A.B.* On the Theory of the Direct Four-fermion Interaction // Sov. Phys. JETP. 1976 V. 44. P. 11.
- [10] *'t Hooft G. and Veltman M.J.G.* One-loop Divergences in the Theory of Gravitation // Annalen Poincaré Phys. Theor. A. 1974. V. 20. P. 69.
- [11] *Robert D. and Smilga A.V.* Supersymmetry vs. Ghosts // J. Math. Phys. 2008. V. 49. P. 042104.
- [12] *Smilga A.V.* Comments on the Dynamics of the Pais-Uhlenbeck Oscillator // SIGMA 2009. V. 5. P. 017.
- [13] *Smilga A.V.* 6D Superconformal Theory as the Theory of Everything // Proc. of Gribov-75 Memorial Workshop on Quarks, Hadrons and Strong interactions. Budapest, May 22–24, 2005. World Scientific, 2006. P. 443 [arXiv:hep-th/0509022].
- [14] *Galperin A.S., Ivanov E.A., Ogievetsky V.I. and Sokatchev E.S.* Harmonic Superspace. Cambridge: Cambridge University Press, 2001.
- [15] *Tseytlin A.A.* On Non-Abelian Generalization of Born-Infeld Action in String Theory // Nucl. Phys. B. 1997. V. 501, P. 41;
Koerber P. and Sevrin A. The Non-Abelian D-brane Effective Action Through Order α'^4 // JHEP 2002. V. 0210. P. 046.
- [16] *Paban S., Sethi S. and Stern M.* Constraints from Extended Supersymmetry in Quantum Mechanics // Nucl. Phys. B. 1998. V. 534 P. 137.
- [17] *Kazama Y. and Muramatsu T.* Fully Off-shell Effective Action and its Supersymmetry in Matrix Theory // Class. Quant. Grav. 2001. V. 18. P. 2277.
- [18] *Nicolai H. and Plefka J.* Supersymmetric Effective Action of Matrix Theory // Phys. Lett. B. 2000. V. 477 P. 309.
- [19] *Becker K., Becker M., Polchinski J. and Tseytlin A.* Higher order graviton scattering in M(atr)ix theory // Phys. Rev. D. 1997. V. 56. P. 3174.

- [20] *Frampton P.H. and Kephart T.W.* Explicit Evaluation of Anomalies in Higher Dimensions // Phys. Rev. Lett. 1983. V. 50. P. 1343;
Alvarez-Gaumé L. and Witten E. Gravitational Anomalies // Nucl. Phys. B. 1984. V. 234. P. 269;
Smilga A.V. Chiral Anomalies in Higher-Derivative Supersymmetric 6D Theories // Phys. Lett. B 2007. V. 647. P. 298.
- [21] *Buchbinder I.L. and Ivanov E.A.* Complete $\mathcal{N} = 4$ Structure of Low-energy Effective Action in $\mathcal{N} = 4$ Super-Yang-Mills Theories // Phys. Lett. B 2002. V. 524. P. 208.
- [22] *Drummond J.M., Heslop P.J., Howe P.S. and Kerstan S.F.* Integral Invariants in $\mathcal{N} = 4$ SYM and the Effective Action for Coincident D-branes // JHEP 2003. V. 0308. P. 016.
- [23] *B.M. Zupnik* Six-dimensional Supergauge Theories in the Harmonic Superspace // Sov. J. Nucl. Phys. 1986. V. 44. P. 512.
- [24] *Ivanov E.A., Smilga A.V. and Zupnik B.M.* Renormalizable Supersymmetric Gauge Theory in Six Dimensions // Nucl. Phys. B. 2005. V. 726. P. 131.
- [25] *Ivanov E.A. and Smilga A.V.* Conformal Properties of Hypermultiplet Actions in Six Dimensions // Phys. Lett. B. 2006. V. 637. P. 374.
- [26] *Howe P.S. and Stelle K.S.* Ultraviolet Divergences in Higher Dimensional Supersymmetric Yang-Mills Theories // Phys. Lett. B. 1984. V. 137. P. 175.
- [27] *Buchbinder I.I. and Pletnev N.G.* Leading Low-energy Effective Actions in the 6D Hypermultiplet Theory on a Vector/Tensor Background // Phys. Lett. B. 2015. V. 744. P. 125.
- [28] *Gross D.J. and Witten E.* Superstring Modifications of Einstein's Equations // Nucl. Phys. B. 1986. V. 277. P. 1.
- [29] *Bergshoeff E., Rakowski M. and Sezgin E.* Higher Derivative Super-Yang-Mills Theories // Phys. Lett. B. 1987. V. 185. P. 371.
- [30] *Bossard G., Howe P.S. and Stelle K.S.* The Ultra-violet Question in Maximally Supersymmetric Field Theories // Gen. Rel. Grav. 2009. V. 41. P. 919.
- [31] *Grisaru M.T.* Two-loop Renormalizability of Supergravity // Phys. Lett. B. 1977. V. 66. P. 75.
- [32] *Deser S., Kay J.H. and Stelle K.S.* Renormalizability Properties of Supergravity // Phys. Rev. Lett. 1977. V. 38. P. 527.
- [33] *Kallosh R.E.* Counterterms in Extended Supergravities // Phys. Lett. B. 1981. V. 99. P. 122;
Howe P.S. and Lindström U. Higher-order Invariants in Extended Supergravity // Nucl. Phys B. 1981. V. 181. P. 487.

- [34] *N. Beisert, Elwang H., Freedman D.Z., Kiermaier M., Morales A. and Stieberger S.* $E_{7(7)}$ Constraints on Counterterms in $\mathcal{N} = 8$ Supergravity // Phys. Lett. B. 2010. V. 694. P. 265.
- [35] *Bossard G., Howe P.S., Stelle K.S. and Vanhove P.* The Vanishing Volume of $D = 4$ Superspace // Class. Quant. Grav. 2011. V. 28. P. 215005.
- [36] *Bern Z., Davies S., Dennen T. and Huang Y.T.* Absence of Three-loop Four-point Divergences in $\mathcal{N} = 4$ Supergravity // Phys. Rev. Lett. 2012. V. 108. P. 201301;
Bern Z., Davies S. and Dennen T. Enhanced Ultraviolet Cancellations in $\mathcal{N} = 5$ Supergravity at Four Loops // Phys. Rev. D. 2014. V. 90. P. 105011.
- [37] *Christensen C.M. Duff M.J., Gibbons G.W. and Roček M.* Vanishing One-loop β Function in Gauged $N > 4$ Supergravity // Phys. Rev. Lett. 1980. V. 45. P. 161;
Curtwright T. Charge Renormalization and High Spin Fields // Phys. Lett. B. 1981. V. 102. P. 17.
- [38] *Weinberg S.* Infrared Photons and Gravitons // Phys. Rev. 1965. V. 140. P. B516;
Donoghue J.F. and Torma T. Infrared Behaviour of Graviton-Graviton Scattering // Phys. Rev. D. 1999. V. 60. P. 024003.